MEH329
DIGITAL SIGNAL PROCESSING

-4-

Discrete Time Systems-2

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Linear Constant-Coefficient Difference Equations (LCCDE)

- We calculate the output (response) of an LTI system using the input and the system’s impulse response
- However, as n gets larger, the convolution sum results in increased computation time and memory requirement
- Systems for which the output can be represented in terms of the input signal’s present and past values and the output signal’s past values reduce these requirement

Linear Constant-Coefficient Difference Equations (LCCDE)

- An important class of LTI systems consists of those systems for which the input x[n] and output y[n] satisfy **Nth-order** eq. form of:

$$\boxed{\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]}$$

Linear Constant-Coefficient Difference Equations

$$a_0 y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_k x[n-m]$$

$$y[n] = \frac{1}{a_0}\left(\sum_{m=0}^{M} b_k x[n-m] - \sum_{k=1}^{N} a_k y[n-k]\right)$$

- If we choose $a_0=1$;

$$\boxed{y[n] = \sum_{m=0}^{M} b_k x[n-m] - \sum_{k=1}^{N} a_k y[n-k]}$$

- All systems cannot be represented in terms of an LCCD

Linear Constant-Coefficient Difference Equations

If the output is not related with the previous output values:

$$y[n] = \sum_{m=0}^{M} b_m x[n-m]$$

As a special case, if the output of the system does not depend on past output values:

$$h[n] = \sum_{m=0}^{M} b_m \delta[n-m]$$

Linear Constant-Coefficient Difference Equations

Sistemin dürtü yanıtı doğrudan giriş değerlerinin katsayıları olarak bulunduğundan LCCDE gösterimikonvolusyon toplamına dönüşmektedir.

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k] x[n-k]$$

- Sistem çıkışının sadece giriş değerlerine bağlı olduğu bu sistemşerde LCCDE ile konvolusyon toplamının sonucu aynıdır.
- Sistemin dürtü yanıtı sonlu sayıda sıfırdan farklı değerlere sahip olduğu için bu tür sistemler yapı itibarıyle FIR sistemlerdir.
- The length of impulse response is **M+1** (FIR)

Linear Constant-Coefficient Difference Equations

- Example:

$$y[n] = x[n] - x[n-1]$$

$$x[n] = \left\{ 2, \underset{\underset{n=0}{\uparrow}}{1}, 0, -1, -2, 1 \right\}$$

$n < -1$: $x[n]=0$ and $y[n]=0$

$n = -1$: $y[-1]=x[-1]-x[-2]=2$

$n = 0$: $y[0]=x[0]-x[-1]=-1$

$n = 1$: $y[1]=x[1]-x[0]=-1$

$n = 2$: $y[2]=x[2]-x[1]=-1$

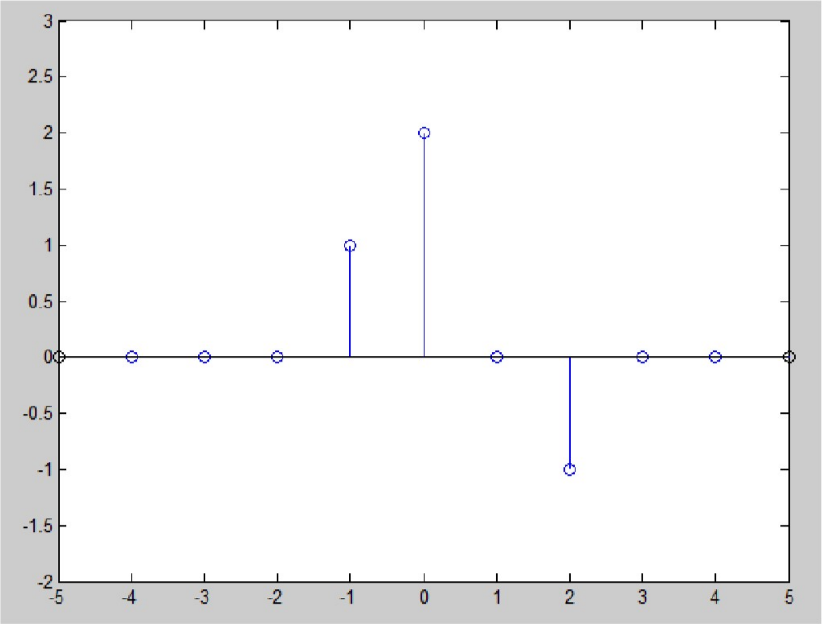
$n = 3$: $y[3]=x[3]-x[2]=-1$

$n = 4$: $y[4]=x[4]-x[3]=3$

$n = 5$: $y[5]=x[5]-x[4]=-1$

$n > 5$: $y[n]=0$

- X[n] y[n]



3
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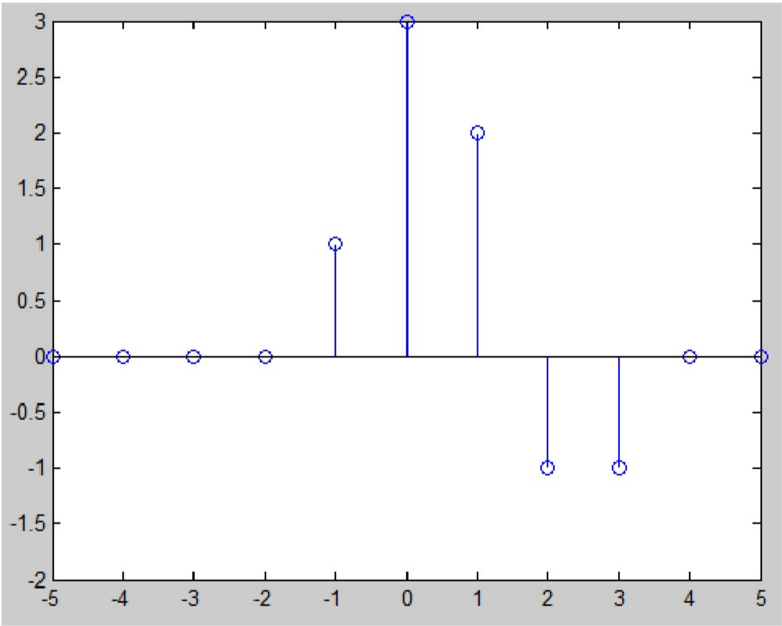

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Linear Constant-Coefficient Difference Equations

- Özyinesiz(Non-Recursive): Bir sistem LCCDE şeklinde tanımlandığında sistem çıkışı girişin o andaki ve eski değerlerine bağlı ise

- Özyineli (Recursive): Bir sistem LCCDE şeklinde tanımlandığında sistem çıkışı girişin o andaki ve eski değerlerine ek olarak çıkışın da eski değerlerine bağlı ise

- Recursive sistemlerde giriş değerleri yanında çıkışın başlangıç koşulları da verilmelidir.

- Başlangıç koşulları sistemin çalışma biçimine doğrudan etkisi bulunduğu için onemlidir.

LCCDE

- Example: Find the output of the accumulator system

$$y[n]=\sum_{k=-\infty}^{n} x[k]$$

$$y[n]=x[n]+\sum_{k=-\infty}^{n-1} x[k]$$

LCCDE representation $$y[n]=x[n]+y[n-1]$$

$$y[n]-y[n-1]=x[n] \longrightarrow$$

$$N=1$$

$$a_0=1$$

$$a_1=-1$$

$$b_0=1$$

Linear Constant-Coefficient Difference Equations

$$x[n] = \left\{ \underset{\substack{\uparrow \\ n=0}}{1}, 0, -1, -2, 1 \right\}$$

- We need an initial value for y

$$y[0] = y[-1] + x[0]$$

- If the system start with y[-1]=0

Linear Constant-Coefficient Difference Equations

$n < 0$: $y[n]=0$

$n = 0$: $y[0]=x[0]+y[-1]=1$

$n = 1$: $y[1]=x[1]+y[0]=1$

$n = 2$: $y[2]=x[2]+y[1]=0$

$n = 3$: $y[3]=x[3]+y[2]=-2$

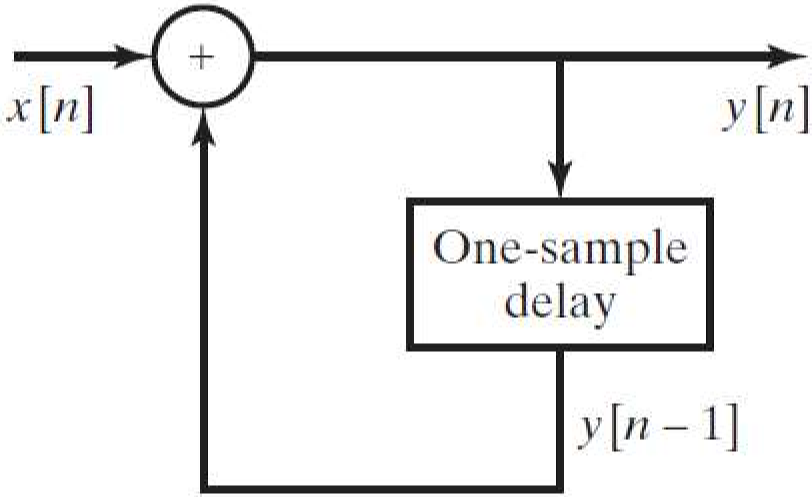
$n = 4$: $y[4]=x[4]+y[3]=-1$

$n = 5$: $y[5]=x[5]+y[4]=-1$

$n > 5$: $y[n]=-1$

Linear Constant-Coefficient Difference Equations

- Block diagram of the accumulator:



Linear Constant-Coefficient Difference Equations

- Example: Moving-average system

$$y[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x[n-k]$$

with $M_1 = 0$

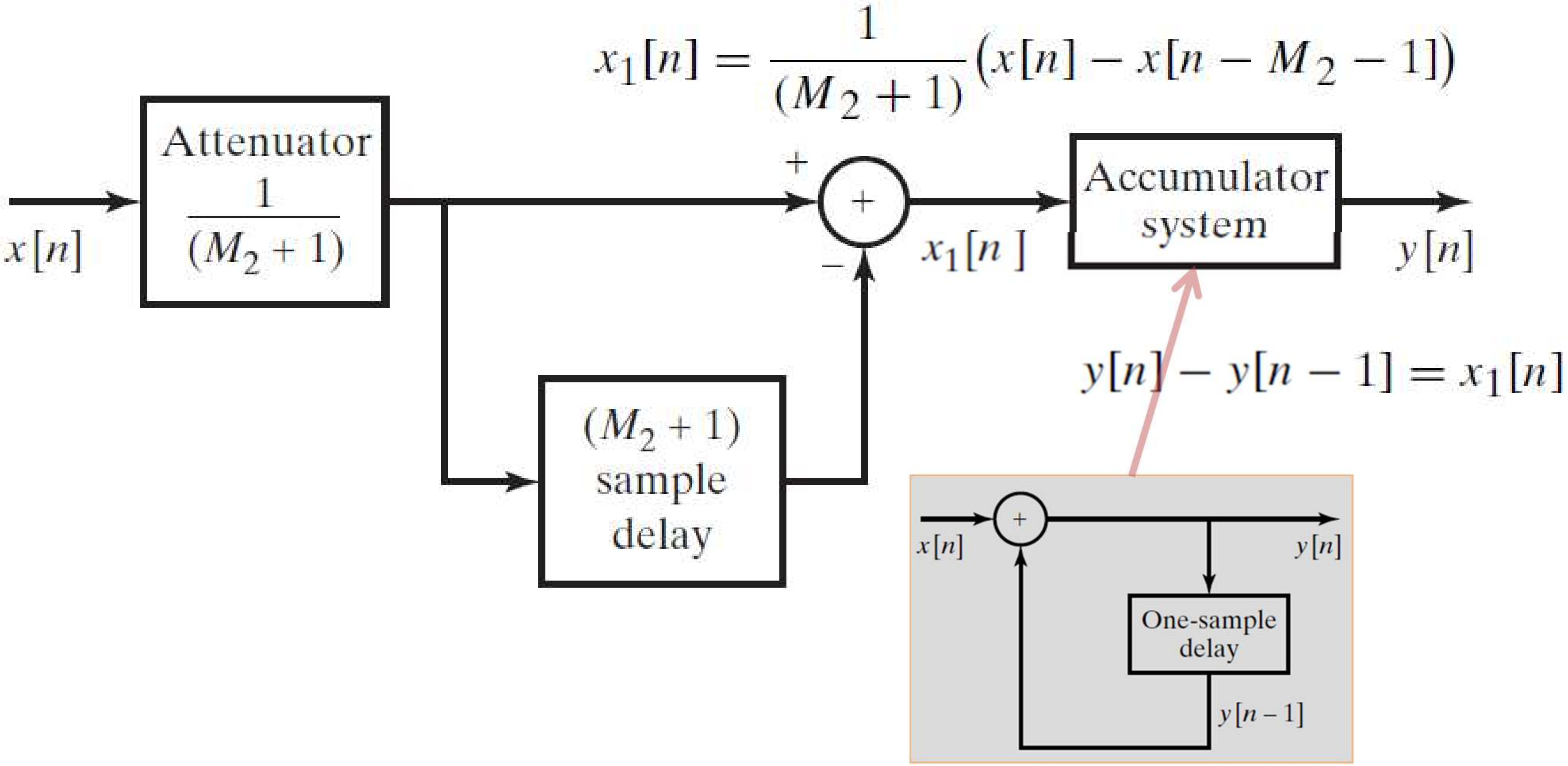
$$y[n] = \frac{1}{(M_2 + 1)} \sum_{k=0}^{M_2} x[n-k]$$

$$h[n] = \frac{1}{(M_2 + 1)} (u[n] - u[n - M_2 - 1])$$

$$y[n] - y[n-1] = \frac{1}{(M_2 + 1)} (x[n] - x[n - M_2 - 1])$$

Linear Constant-Coefficient Difference Equations

- Block diagram of the moving-average system:

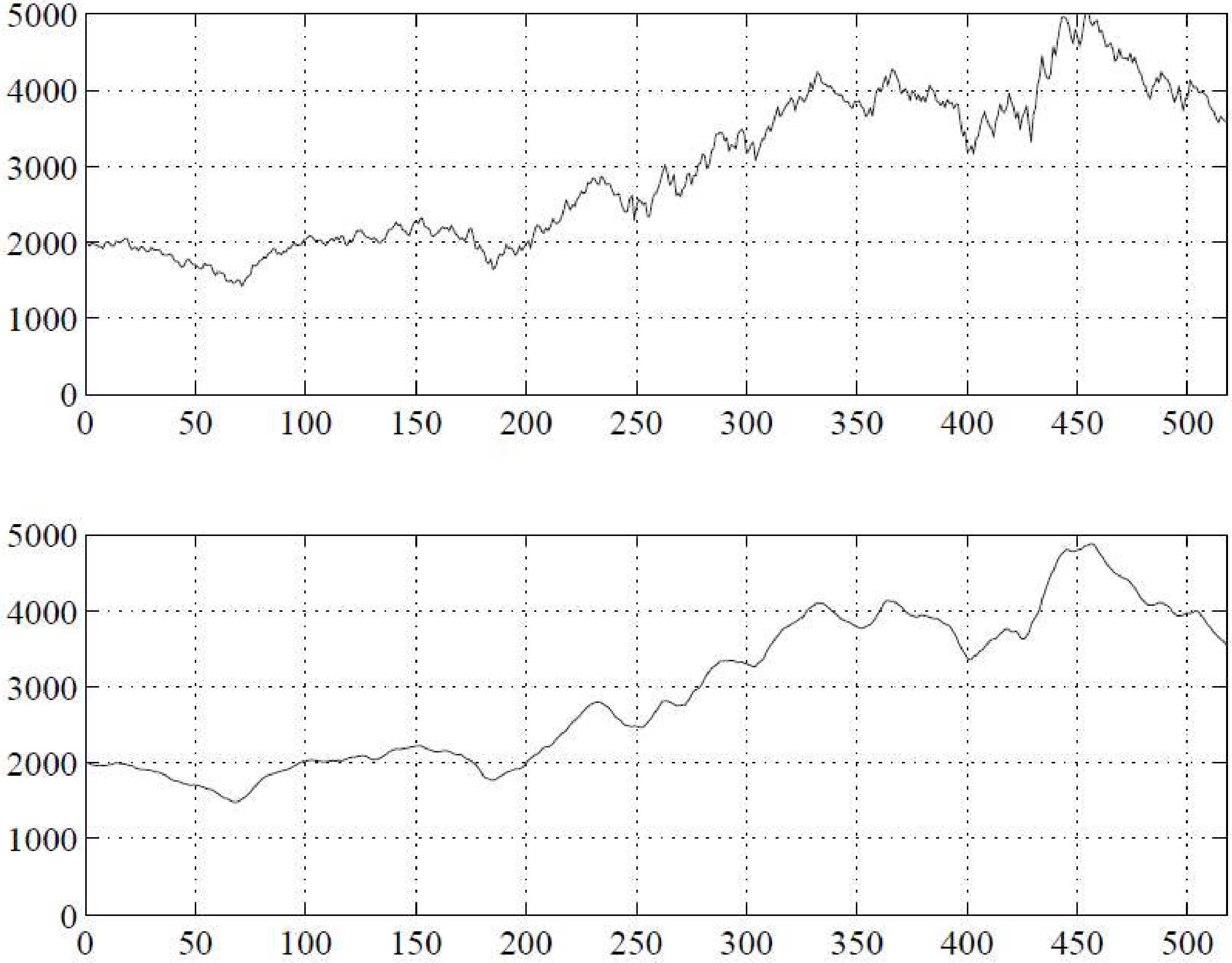


$$x_1[n] = \frac{1}{(M_2+1)}(x[n] - x[n - M_2 - 1])$$

$$y[n] - y[n-1] = x_1[n]$$

Linear Constant-Coefficient Difference Equations

- Effect of moving-average system



Linear Constant-Coefficient Difference Equations

- Example: $y[n] = ay[n-1] + x[n]$

$$x[n] = b\delta[n] \;,\; y[-1] = 1$$

$$\begin{array}{ll} y[0] = a + b & y[-2] = a^{-1} \\ y[1] = a^{2} + ab & y[-3] = a^{-2} \\ y[2] = a^{3} + a^{2}b & y[-4] = a^{-3} \\ y[3] = a^{4} + a^{3}b & y[-5] = a^{-4} \\ \vdots & \vdots \end{array}$$

$$y[n] = a^{n+1} + a^{n}b \;,\; n \geq 0 \qquad y[n] = a^{n+1} \;,\; n < 0$$

Linear Constant-Coefficient Difference Equations

$$y[n] = a^{n+1} + a^n b \;,\; n \geq 0 \qquad\qquad y[n] = a^{n+1} \;,\; n < 0$$

$$\boxed{y[n] = a^{n+1} + a^n b u[n]}$$

- If b=0 → x[n]=0, but $y[n]=a^{n+1}$ is not equal to zero.
- Therefore, scaling the input with zero is not gives zero output (the system is not linear).

Linear Constant-Coefficient Difference Equations

- For the shifted input

$$x_1[n] = x[n - n_d] = b\delta[n - n_d]$$

$$y_1[n] = a^{n+1} + a^{n-n_d} bu[n - n_d]$$

$$y_1[n] \neq y[n - n_d]$$

- the system is time variant.

Linear Constant-Coefficient Difference Equations

- If y[-1]=0 is given

$$y[n] = a^n b u[n]$$

- The system is linear and time invariant in this case (evaluate with $x[n] = b\delta[n-1]$).
- NOTE: Initial conditions of a LCCDE for systems affect the characteristics directly!
- In general, x[n]=0, n<0 and initial conditions are chosen as zero for the causal LTI systems.

Linear Constant-Coefficient Difference Equations

- Alternative representation for the accumulator $y[n] = ay[n-1] + x[n]$

$$y[0] = ay[-1] + x[0]$$

$$y[1] = ay[0] + x[1] = a(ay[-1] + x[0]) + x[1] = a^2 y[-1] + ax[0] + x[1]$$

$$y[2] = ay[1] + x[2] = a^3 y[-1] + a^2 x[0] + ax[1] + x[2]$$

$$\vdots$$

$$y[n] = a^{n+1} y[-1] + \sum_{k=0}^{n} a^k x[n-k], \quad n \geq 0$$

Response to the initial conditions

Response to the input signal x[n]

Linear Constant-Coefficient Difference Equations

- If the system initially relaxed at time n=0

y[-1]=0

- Thus a recursive system is relaxed if it starts with zero initial conditions.
- We say that the system is at “**zero state**” in this case.
- The response of the system is called as

“ZERO STATE RESPONSE”

$$\boxed{y_{zs}[n]}$$

Linear Constant-Coefficient Difference Equations

$$\boxed{y_{zs}\left[n\right]=\sum_{k=0}^{n}a^{k}x\left[n-k\right]\ ,\ \ n\geq 0}$$

$$h\left[n\right]=a^{n}u\left[n\right]$$ (IIR, causal)

Linear Constant-Coefficient Difference Equations

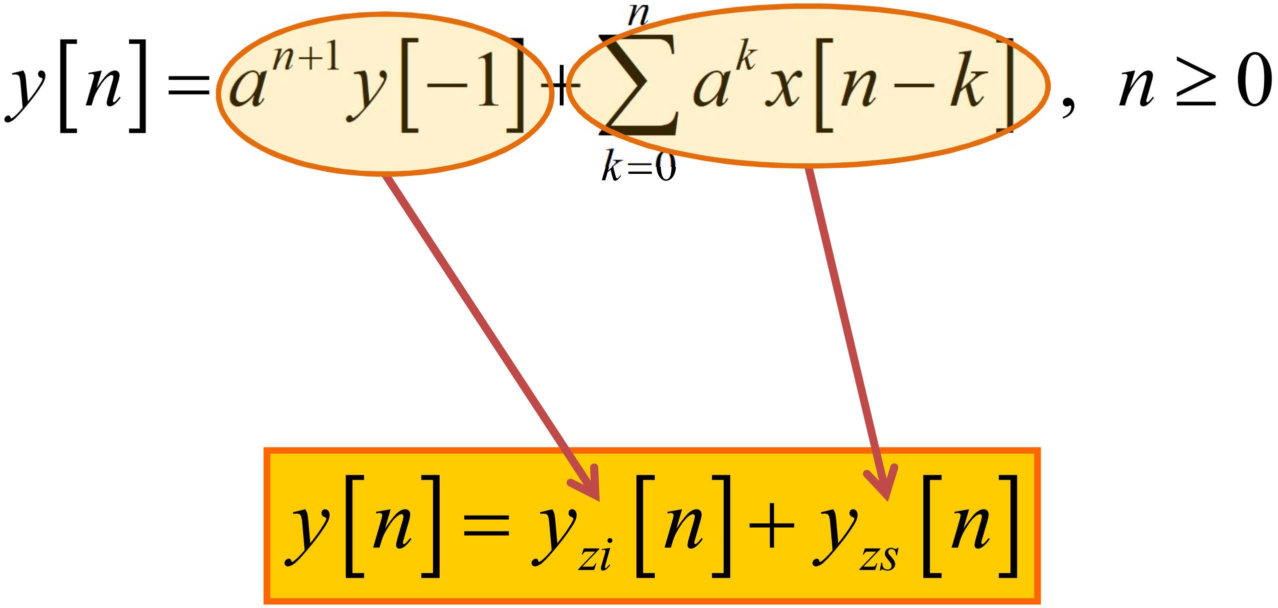
- Suppose that the system is initially nonrelaxed and x[n]=0 for all n.
- Then the output of the system with zero input is called the

“ZERO INPUT RESPONSE”

$$\boxed{y_{zi}[n]}$$

$$\boxed{y_{zi}[n]=a^{n+1}y[-1]}$$

Linear Constant-Coefficient Difference Equations



$$y[n] = a^{n+1} y[-1] + \sum_{k=0}^{n} a^k x[n-k], \quad n \geq 0$$

$$\boxed{y[n] = y_{zi}[n] + y_{zs}[n]}$$